The multihistory approach to the time-travel paradoxes of General Relativity: mathematical analysis of a toy model

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With a mathematical eye to Matt Visser's multihistory approach to the time-travel-paradoxes of General Relativity, a non relativistic toy model is analyzed in order of characterizing the conditions in which, in such a toy model, causation occurs.

Contents

I. CAUSATION VERSUS CHRONOLOGY VIOLATIONS IN GENERAL RELATIVITY

The only consistent definition of the concept of causality is based on the Cauchy-Kowalewski theorem (see for instance the section 1.7 of [1]) stating that the initial value problem of a broad class of (partial) differential equations has one and only one solution: in that case one defines causation as the inter-relation existing between the initial value and the later predicted value of the involved quantity [1].

Let us remark that, in the general case, the evolution parameter involved in the Cauchy problem doesn't necessarily coincide with the physical time.

Often it is is implicitly assumed the following:

Conjecture I.1

Conjecture of Causation:

The (partial) differential equations expressing the Laws of Nature have to admit, thanks to the Cauchy-Kowalewski theorem, a well-defined initial value problem.

Indeed this is the situation to which we were used since the constraint involved in the conjecture I.1 is satisfied by both Newtonian Mechanics and Special Relativity.

It is even satisfied by Quantum Mechanics where, beside the nondeterministic reduction process, the evolution of a system non subjected to measurements is ruled by the Schrödinger equation that of course admits a well-defined initial value problem.

One of the philosophical peculiarities of General Relativity [3], [4] is that, without assuming ad hoc suppletive axioms in order to avoid such a situation, it violates ConjectureI.1: Einstein's equation $R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi T_{ab}$ admits solutions (M, g_{ab}) that are not globally-hyperbolic (i.e. that don't admit a Cauchy surface, namely a closed achronal set S whose *domain of dependence* is the whole spacetime $D(S) = M$):

in this case for every choice of a three-dimensional C^{∞} submanifold S the predictability of the structure of spacetime (and hence the causation) allowed from the initial value problem on S is limited to the interior of the Cauchy-horizon of S:

$$H_{Cauchy}(S) \; := \; \partial D(S) \tag{1.1}$$

The fact that in such a situation the scalar wave equation may have a well-defined Cauchy problem [5] is a very little consolation.

An additional peculiarity of General Relativity concerns chronology violations:

given a time-orientable space-time (M, g_{ab}):

Definition I.1

chronology violating set of (M, g_{ab})*:*

$$V_{chronology}(M, g_{ab}) \; := \; \cup_{p \in M} I^{+}(p) \cap I^{-}(p) \tag{1.2}$$

A curious feature of General Relativity is that there exist solutions (M, g_{ab}) of Einstein's equation such that $V_{chronology}(M, g_{ab}) \; \neq \; \emptyset$.

Given such a solution and two points $p_1, p_2 \in M$:

Definition I.2

chronological relation:

$$p_1 \; \preceq_{chronological} \; p_2 \; := p_2 \in I^{+}(p_1) \vee p_1 = p_2 \tag{1.3}$$

Then:

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[1] Contrary to the typical philosophical attitude of complicating simple things (see for instance [2]) as to the definition of predictability we assume that the initial condition may be known with exactness and we don't care about the possible incomputability of the map expressing the final state (the effect) as a function of the initial state (the cause).

Proposition I.1

- $\preceq_{chronological}$ is a partial ordering relation over $M - V_{chronology}(M, g_{ab})$
- $\preceq_{chronological}$ is a preordering over $V_{chronology}(M, g_{ab})$ but is not a partial ordering
- $$\frac{V_{chronology}(M, g_{ab})}{\sim_{\preceq_{chronological}}} = \{I^{+}(p) \cap I^{-}(p) \; p \in V_{chronology}(M, g_{ab})\} \tag{1.4}$$

PROOF:

Let us observe, first of all, that definitionI.2 differs slightly from the usual way through which the chronological relation (traditionally denoted by <<) is defined in the literature (see for instance the section 2.1 of [6], [7] , the section3.2 "Causality Theory of Space-times" of [8] and [9]) that would correspond to our $p_1 \prec_{chronological} p_2 \; := p_1 \preceq_{chronological} p_2 \wedge p_1 \neq p_2$ [2]; then:

$$p \preceq_{chronological} p \;\; \forall p \in M \tag{1.5}$$

Demanding to the appendix A for the involved notions concerning preorderings, the thesis follows by the fact that [3] $V_{chronology}(M, g_{ab})$ is the disjoint union of sets of the form $I^{+}(p) \cap I^{-}(p)$ for $p \in V_{chronology}(M, g_{ab})$ and that:

$$p_1 \sim_{\preceq_{chronological}} p_2 \;\; \forall p_1, p_2 \in I^{+}(p) \cap I^{-}(p), \forall p \in V_{chronology}(M, g_{ab}) \tag{1.6}$$

■

The situation delineated by propositionI.1 is often said to generate the so called time travel paradoxes.
These paradoxes can be divided in two classes:

- *consistency paradoxes* involving the effects of the changes of the past (epitomized by the celebrated Grandfather Paradox in which a time-traveller goes back in the past and prevents the meeting of his grandfather and his grandmother)
- *bootstrap paradoxes* involving the presence of loops in which the source of the production of some information disappears (as an example let us suppose that Einstein learnt Relativity Theory from [3], [4] given to him by a time-traveller gone back to 1904).

Remark I.1

As correctly stated in the section 6.4 "Causality conditions" of [3] these paradoxes occur only if one assumes a simple notion of human free-will.

The existence of human free will is therein claimed to be a corner stone of the Philosophy of Science underlying the Scientific Method based on the assumption that one is free to perform any experiment.

We don't agree with such a claim since, in our opinion, the hypothetical assumption that we are determined to make the experiments we perform is not incompatible with such a Philosophy of Science.

Furthermore, at first sight, human free will would seem to be incompatible with the determinism of General Relativity.

A deeper investigation about this claim requires:

- a precise definition of both the concepts of determinism and free-will
- the analysis whether, according to the assumed definitions of both the terms, determinism and free will are compatible (the position of Compatibilists) or not (the position of Incompatibilists) [3]

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[2] the chronological relation may be in seen in some contexts as a conceptually more fundamental mathematical structure than the metric (from which, anyway, its definition obviously depends); an instance is given by the definition of causal boundaries [10] where other kinds of non Hausdorffness, differing from those discussed in this paper, occur.

[3] We are using here the standard philosophical terminology [11]; such a terminology introduces also the set of the Libertarians defined as the subset of the Incompatibilists consisting in those believing in the existence of the free will while denying the existence of determinism. Apart from a general skeptisism about the whole philosophical approach to these themes, we regret the adoption of a term like libertarian, to which a positive ethical connotation is usually ascribed, to denote a conceptual position whose truth or falsity should be neutrally decided analyzing the features of the Physical Theories describing Nature.

The determinism of a physical theory may be defined as the condition that there exist dynamical equations (suitable (partial) differential equations) governing the evolution of any closed system.

Let us remark that, contrary to Laplace's classical definition (see the celebrated second chapter "Concerning probability" of [12]), such a definition of determinism doesn't assume the Conjecture I.1.

The simple notion of human free-will above mentioned consists in the assumption that the true scientific theory describing human mind is deterministic.

Obviously such a simple notion of free-will leads, by definition, to the incompatibilist thesis.

There exist, anyway, more refined definitions of free-will that, once assumed, make the compatibilistic thesis consistent.

The problem of the so called time-travel paradoxes has been faced by the scientific community in different ways (see the fourth part "Time Travel" of [13] as well as [14]):

1. adding to General Relativity some ad hoc axiom precluding the physical possibility of causal loops (such as the strong form of Penrose's Cosmic Censorship Conjecture)

2. appealing to consistency conditions (such as in Novikov's Consistency Conjecture) requiring that causal loops, though allowing causal influence on the past, don't allow alteration of the past

3. arguing that the problem is removed at a quantum level (such as in Hawking's Chronology Protecting Conjecture stating that the classical possibilities to implement time-travels are destroyed by quantum effects)

4. arguing that the so called time-travel paradoxes are only apparent and may be bypassed in a mathematical consistent way

We have nor the knowledge neither the competence to take sides about such a subtle issue.

In our opinion, anyway, of particular interest from a mathematical viewpoint is the particular approach of the fourth kind according to which the so called time-travel paradoxes are bypassed with the removal of the assumption that space-time, as a topological space, has to be Hausdorff (and hence allowing Universe's bifurcations in multiple histories; see the section 19.1 "The radical rewrite conjecture" of [13] based on previous remarks by Robert Geroch and Roger Penrose [15] as well as by Petr Hajicek [16] [4]) in that it allows an interesting investigation about the topological structure of the evolution parameter's space required in order that, eventually under suitable consistency conditions, conjecture I.1 holds.

[4] As it has happened many times, the idea underlying such an approach has appeared first in Science-Fiction's literature than in Science's literature: it is for this reason that we suggest the lecture of the parts of [17] concerning the logical analysis of time-travel issues in the Science Fiction's literature.

II. VISSER SPLITTINGS

Given a topological space (see appendix B) $(X, \mathcal{T})$, a subset of its $\Omega \subset X$ such that $\partial\Omega \neq \emptyset$ and a natural number $n \in \mathbb{N} : n \geq 2$:

Definition II.1

Visser n-splitting of X through Ω*:*

$$Split(X, \Omega, n) := [X - \bar{\Omega}] \cup [\cup_{i=1}^{n} \bar{\Omega}_i] \tag{2.1}$$

where $\Omega_1, \cdots, \Omega_n$ are n disjoint copies of Ω.

Definition II.2

natural topology of $Split(X, \Omega, n)$*:*

the topology $\tilde{\mathcal{T}}(X, \Omega, n)$ having the following basis: any open set in $[X - \bar{\Omega}] \cup \bar{\Omega}_i, i = 1, \cdots, n$ is an open set of $\tilde{\mathcal{T}}(X, \Omega, n)$.

From here and beyond we will assume that any Visser splitting is endowed with its natural topology.
It is important to observe that:

Proposition II.1

non Hausdorffness of Visser splittings:

the topological space $(Split(X, \Omega, n), \tilde{\mathcal{T}}(X, \Omega, n))$ is not Hausdorff

PROOF:

Given $i, j = 1, \cdots, n : i \neq j$, let us consider two points $x_i \in \partial\Omega_i, x_j \in \partial\Omega_j$ such that x_i and x_j are copies of the same element $x \in \partial\Omega$; the definition II.2 implies that $x_i \curlyvee x_j$. $\blacksquare$

Let us now suppose that $\preceq$ is a partial ordering over X (see appendix A). Given $x, y \in Split(X, \Omega, n)$:

Definition II.3

$$x \tilde{\preceq} y := x \preceq y \tag{2.2}$$

Then:

Proposition II.2

- $\tilde{\preceq}$ is a preordering over $Split(X, \Omega, n)$
- $\tilde{\preceq}$ is not a partial ordering over $Split(X, \Omega, n)$

PROOF:

The reflexive and the transitive property of $\tilde{\preceq}$ may be immediately inferred by the corresponding properties of $\preceq$.

$\tilde{\preceq}$ doesn't satisfy, anyway, the antisymmetric property since given $x \in \Omega$ and considered the corresponding copies $x_i \in \Omega_i$ $i = 1, \cdots, n$:

$$(x_i \tilde{\preceq} x_j \wedge x_j \tilde{\preceq} x_i \nRightarrow x_i = x_j) \;\; i, j = 1, \cdots, n : i \neq j \tag{2.3}$$

$\blacksquare$

Remark II.1

Gordon Mc Cabe has recently [18] introduced the following alternative to Visser splitting:

Definition II.4

Mc Cabe n-splitting of X through Ω:

$$Split_{Mc\,Cabe}(X,\Omega,n) \;:=\; \frac{Split(X,\Omega,n)}{\sim_{Mc\,Cabe}} \tag{2.4}$$

where $\sim_{Mc\,Cabe}$ is the equivalence relation over $Split(X,\Omega,n)$ defined by:

$$x_i \sim_{Mc\,Cabe} x_j \;\Leftrightarrow\; x_i \in \partial\Omega_i, x_j \in \partial\Omega_j \wedge \; x_i \text{ and } x_j \text{ are copies of the same element } x \in \partial\Omega \;\; i,j = 1,\cdots,n \;:\; i \neq j \tag{2.5}$$

Definition II.5

natural topology of $Split_{Mc\,Cabe}(X,\Omega,n)$:

$$\tilde{\mathcal{T}}_{Mc\,Cabe}(X,\Omega,n) \;:=\; \frac{\tilde{\mathcal{T}}(X,\Omega,n)}{\sim_{Mc\,Cabe}} \tag{2.6}$$

Then:

Proposition II.3

$$\sim_{Mc\,Cabe} \;\text{ is Hausdorff} \tag{2.7}$$

PROOF:

Since:

$$x \curlyvee y \;\Rightarrow\; x \sim_{Mc\,Cabe} y \;\; \forall x,y \in Split(X,\Omega,n) \tag{2.8}$$

the thesis follows ■

Actually Mc Cabe's formalism is nothing but an application of the strategy indicated by Geroch in the exercise 177 of the 27^{th} section "Continuous Mappings" of [19] :

"make equivalent as few points as necessary to get an Hausdorff quotient space"

For this reason we won't adopt Mc Cabe splittings in this paper.

III. VISSER SPLITTINGS OF TIME AND INITIAL VALUE PROBLEMS: A TOY MODEL

We will consider for simplicity from here and beyond a particular toy model consisting of a classical non-relativistic dynamical system having as configuration space the real line $(\mathbb{R}, T_{natural}(\mathbb{R}))$ and whose evolution parameter t (that in this particular case will hence coincide with the physical absolute time) takes values on some Visser splitting of the real line $(Split(\mathbb{R}, \Omega, n), \tilde{\mathcal{T}}(\mathbb{R}, \Omega, n))$.

The usual linear ordering $\leq$ over $\mathbb{R}$ induces, by proposition II.2, the preordering $\tilde{\leq}$ over $Split(\mathbb{R}, \Omega, n)$ that, in according to the explained underlying physical interpretation, we will denote by $\preceq_{chronological}$.

Given a map $x : (Split(\mathbb{R}, \Omega, n), \tilde{\mathcal{T}}(\mathbb{R}, \Omega, n)) \mapsto (\mathbb{R}, +, \cdot)$, according to the analysis performed in the appendix C it is well defined the concept of time derivative of $x(t)$ that we will denote by $\dot{x}(t)$.

Given a map $f \in C^{\infty}(\mathbb{R})$, a point $t_{in} \in Split(\mathbb{R}, \Omega, n)$ and a real number $x_{in} \in \mathbb{R}$ we will investigate under which conditions on Ω the initial value problem:

$$\dot{x} \ = \ f(x) \tag{3.1}$$

$$x(t_{in}) \ = \ x_{in} \tag{3.2}$$

is well-posed, i.e. it admits one and only one solution $x(t) : (Split(\mathbb{R}, \Omega, n), \tilde{\mathcal{T}}(\mathbb{R}, \Omega, n)) \mapsto (\mathbb{R}, +, \cdot)$.

Let us start considering the simplest case $\Omega := \{0\}, n := 2$.

Let us start from the case $t_{in} \notin \{0_1, 0_2\}$.

Called $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ the solution of the Cauchy problem eq. 3.1, eq. 3.2 for ordinary time $t \in \mathbb{R}$:

Proposition III.1

HP:

$$t_{in} \in Split(\mathbb{R}, \{0\}, 2) - \{0_1, 0_2\} \tag{3.3}$$

TH:

The Cauchy problem eq. 3.1, eq. 3.2 for $t \in Split(\mathbb{R}, \{0\}, 2\}$ is well-defined, its unique solution being the map $\tilde{\tilde{x}}(t) : Split(\mathbb{R}, \{0\}, 2) \mapsto \mathbb{R}$:

$$\tilde{\tilde{x}}(t) \ := \ \tilde{x}(t) \ \forall t \in \mathbb{R} - \{0\} \tag{3.4}$$

$$\tilde{\tilde{x}}(0_1) \ := \ \tilde{x}(0) \tag{3.5}$$

$$\tilde{\tilde{x}}(0_2) \ := \ \tilde{x}(0) \tag{3.6}$$

PROOF:

The thesis immediately follows by the definition C.3 and the structure of the topology $\tilde{\mathcal{T}}(\mathbb{R}, \{0\}, 2)$. ■

Let us now suppose that $t_{in} := 0_1$ and let $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem:

$$\dot{x} \ = \ f(x) \tag{3.7}$$

$$x(0) \ = \ x_{in} \tag{3.8}$$

Then:

Proposition III.2

HP:

$$t_{in} := 0_1 \tag{3.9}$$

TH:

The Cauchy problem eq. 3.1, eq. 3.2 for $t \in Split(\mathbb{R}, \{0\}, 2\}$ is well-defined, its unique solution being the map $\tilde{\tilde{x}}(t) : Split(\mathbb{R}, \{0\}, 2) \mapsto \mathbb{R}$:

$$\tilde{\tilde{x}}(t) := \tilde{x}(t) \ \forall t \in \mathbb{R} - \{0\} \tag{3.10}$$

$$\tilde{\tilde{x}}(0_1) := \tilde{x}(0) \tag{3.11}$$

$$\tilde{\tilde{x}}(0_2) := \tilde{x}(0) \tag{3.12}$$

PROOF:

It follows from the definition C.3 and the structure of the topology $\tilde{\mathcal{T}}(\mathbb{R}, \{0\}, 2)$. ■

The generalization to $n \in \mathbb{N} : n > 2$ is straightforward.

Let us now consider the case in which $\Omega := [0, +\infty), n := 2$.
If $t_{in} \in (-\infty, 0)$ let $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem eq. 3.1, eq. 3.2 for ordinary time $t \in \mathbb{R}$. Then:

Proposition III.3

HP:

$$t_{in} \in (-\infty, 0) \tag{3.13}$$

TH:

The Cauchy problem eq. 3.1, eq. 3.2 for $t \in Split(\mathbb{R}, [0, +\infty), 2)$ is well-defined, its unique solution being the map $\tilde{\tilde{x}}(t) : Split(\mathbb{R}, [0, +\infty), 2) \mapsto \mathbb{R}$:

$$\tilde{\tilde{x}}(t) := \tilde{x}(t) \ \forall t \in (-\infty, 0) \tag{3.14}$$

$$\tilde{\tilde{x}}(t_1) := \tilde{x}(t = t_1) \ \ \forall t_1 \in [0_1, +\infty_1) \tag{3.15}$$

$$\tilde{\tilde{x}}(t_2) := \tilde{x}(t = t_2) \ \ \forall t_2 \in [0_2, +\infty_2) \tag{3.16}$$

PROOF:

The thesis immediately follows by the definition C.3 and the structure of the topology $\tilde{\mathcal{T}}(\mathbb{R},\{0\},2)$. ∎

Given $a,b \in \mathbb{R}$ let us consider the generalized Cauchy problem:

$$\dot{x} = f(x) \tag{3.17}$$

$$x(0_1) = a \tag{3.18}$$

$$x(0_2) = b \tag{3.19}$$

Given an arbitrary $r \in \mathbb{R}$ let $\tilde{x}(t)_r : \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem:

$$\dot{x} = f(x) \tag{3.20}$$

$$x(0) = r \tag{3.21}$$

for ordinary time $t \in \mathbb{R}$.

Then:

Proposition III.4

The generalized Cauchy problem eq.3.17, eq.3.18, eq. 3.19 for $t \in Split(\mathbb{R},[0,+\infty),2)$ is well-defined if and only if $a=b$; in this case the unique solution is the map $\tilde{\tilde{x}}(t) : Split(\mathbb{R},[0,+\infty),2) \mapsto \mathbb{R}$ such that:

$$\tilde{\tilde{x}}(t) := \tilde{x}(t)_{a=b} \ \ \forall t \in (-\infty,0) \tag{3.22}$$

$$\tilde{\tilde{x}}(t_1) := \tilde{x}(t=t_1)_{a=b} \ \ \forall t_1 \in [0_1,+\infty_1) \tag{3.23}$$

$$\tilde{\tilde{x}}(t_2) := \tilde{x}(t=t_2)_{a=b} \ \ \forall t_2 \in [0_2,+\infty_2) \tag{3.24}$$

PROOF:

The retrodiction to $(-\infty,0)$ is possible if and only if:

$$\tilde{x}(t)_a = \tilde{x}(t)_b \ \ \forall t \in (-\infty,0) \tag{3.25}$$

and hence if and only if $a=b$. ∎

Let us now suppose that $t_{in} \in [0_1,+\infty_1)$ and let $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ be the solution of the Cauchy problem eq. 3.1, eq. 3.2 for ordinary time $t \in \mathbb{R}$. Then:

Proposition III.5

HP:

$$t_{in} \in [0_1,+\infty_1) \tag{3.26}$$

TH:

The Cauchy problem eq. 3.1, eq. 3.2 for $t \in Split(\mathbb{R},[0,+\infty),2)$ is well-defined, its unique solution being the map $\tilde{\tilde{x}}(t) : Split(\mathbb{R},[0,+\infty),2) \to \mathbb{R}$:

$$\tilde{\tilde{x}}(t) \ := \ \tilde{x}(t) \ \ \forall t \in (-\infty,0) \tag{3.27}$$

$$\tilde{\tilde{x}}(t_1) \ := \ \tilde{x}(t=t_1) \ \ \forall t_1 \in [0_1,+\infty_1) \tag{3.28}$$

$$\tilde{\tilde{x}}(t_2) \ := \ \tilde{x}(t=t_2) \ \ \forall t_2 \in [0_2,+\infty_2) \tag{3.29}$$

PROOF:

The solution on the first branch $[0_1,+\infty_1)$ allows the retrodiction on $(-\infty,0)$ from which the prediction on $[0_2,+\infty_2)$ can be derived. ∎

IV. THE TOY MODEL WITH A TREE-LIKE TOPOLOGICAL STRUCTURE OF TIME

The situation discussed in the previous section may be easily generalized to the case of multiple Visser splitting of time:

given $n \in \mathbb{N}_+$, $t_1, \cdots, t_n \in \mathbb{R}$ such that $t_1 < t_2 < \cdots < t_n$, n natural numbers greater or equal than two $b_1, b_2, \cdots, b_n \in \mathbb{N} : b_i \geq 2 \;\; i = 1, \cdots, n$ and $n-1$ natural numbers $i_1 \in \{1, 2, \cdots, b_1\}, \cdots, i_{n-1} \in \{1, 2, \cdots, b_n\}$:

Definition IV.1

temporal tree:

$$tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n) := Split(\cdots Split(Split(\mathbb{R}, [t_1, +\infty), b_1), [t_{2,(i_1)}, +\infty_{(i_1)}), b_2) \cdots, [t_{n,(i_{n-1})}, +\infty_{(i_{n-1})}), b_n) \quad (4.1)$$

Remark IV.1

It important not to make confusion between the two different kind of labels: those denoting the time ordering and those denoting the different branches; to avoid confusion we will denote this second kind of label enclosing it between brackets.

So the first time splitting occurs at t_1 of which we will have b_1 copies that we will denote by $t_{1,(1)}, \cdots, t_{1,(b_1)}$ and so on.

Remark IV.2

Let us remark that the iterated adoption of the definition II.2 induces a topology on $tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)$ that we will denote by $\mathcal{T}(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)$.

Remark IV.3

The name of definition IV.1 may be someway misleading and has to be managed carefully.

Its origin arises observing that introduced the following:

Definition IV.2

graph of $tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)$:

the oriented graph [20] $graph[tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)]$ obtained by $tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)$ replacing each element $[t_{i-1,(j)} \cdots, t_{i,(j)}) \cup \cup_{k=1}^{b_i} [t_{i,(k)}, t_{i+1,(k)})$ with a vertex t_i having one entering edge corresponding to the interval $[t_{i-1,(j)} \cdots, t_{i,(j)})$ and having b_i exiting edges corresponding to the intervals $[t_{i,(1)}, t_{i+1,(1)}), \cdots, [t_{i,(b_i)}, t_{i+1,(b_i)})$ (where we have denoted by $t_{(0)} \in (-\infty, t_1)$ the time parameter before the first splitting)

$graph[tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)]$ is indeed a tree.

It is important, anyway, to remark that $graph[tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)]$ is only a diagrammatic way of representing the different mathematical object $tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)$ that is not a graph and hence is not, in particular, a tree.

Given a map $f \in C^{\infty}(\mathbb{R})$, an initial time $t_{in} \in tree(t_1, b_1; \cdots; t_n, b_n)$ and a real number $x_{in} \in \mathbb{R}$:

Proposition IV.1

The Cauchy problem:

$$\dot{x} = f(x) \quad (4.2)$$

$$x(t_{in}) = x_{in} \quad (4.3)$$

is well-posed i.e. it admits one and only one solution

$\tilde{x}(t) : (tree(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n), \mathcal{T}(t_1, b_1; t_{2,(i_1)}, b_2; \cdots; t_{n,(i_{n-1})}, b_n)) \mapsto (\mathbb{R}, +, \cdot)$

PROOF:

The thesis immediately follows by multiple application of Proposition III.3 and Proposition III.5. ■

Given $i \in \{1, \cdots, n\}$ and $a_1, \cdots a_{b_i} \in \mathbb{R}$ let us now consider the generalized Cauchy problem:

$$\dot{x} = f(x) \tag{4.4}$$

$$x(t_{i,(1)}) = a_1, \cdots, x(t_{i,(b_i)}) = a_{b_i} \tag{4.5}$$

Then:

Proposition IV.2

The generalized Cauchy problem of eq. 4.4, eq. 4.5 is well-posed if and only if $a_1 = \cdots = a_{b_i}$

PROOF:

The thesis immediately follows by multiple application of Proposition III.4. ■

V. THE TOY MODEL WITH A LESS TRIVIAL TOPOLOGICAL STRUCTURE OF TIME

Given the Visser splitting $Split(\mathbb{R}, (-\infty, 0], 2)$, a map $f \in C^{\infty}(\mathbb{R})$, $t_{in} \in (-\infty, 0]$ and $a_1, a_2 \in \mathbb{R}$ let us consider the generalized Cauchy problem:

$$\dot{x} = f(x) \tag{5.1}$$

$$x(t_{in,(1)}) = a_1 \tag{5.2}$$

$$x(t_{in,(2)}) = a_2 \tag{5.3}$$

As usual we will say that the Cauchy problem of eq. 5.1, eq. 5.2 and eq. 5.3 is well-posed if and only if it has one and only one solution $\tilde{x} : (Split(\mathbb{R}, (-\infty, 0], 2), \tilde{\mathcal{T}}(\mathbb{R}, (-\infty, 0], 2)) \mapsto (\mathbb{R}, +, \cdot)$.

Then:

Proposition V.1

The Cauchy problem of eq. 5.1, eq. 5.2 and eq. 5.3 is well-posed if and only if $a_1 = a_2$.

PROOF:

The condition $a_1 = a_2$ is necessary and sufficient for the merging of the solutions in the two branches $(-\infty_1, 0_1]$ and $(-\infty_2, 0_2]$ in $(0, +\infty)$. ■

We can now combine the Visser splittings of the form $Split(\mathbb{R}, [t_i, +\infty), n)$ and $Split(\mathbb{R}, [-\infty, t_i), n)$ to obtain more intricate topological structures of time:

given $n \in \mathbb{N}_+$, $t_1, \cdots, t_n \in \mathbb{R}$ such that $t_1 < t_2 < \cdots < t_n$, n natural numbers greater or equal than two $b_1, b_2, \cdots, b_n \in \mathbb{N} : b_i \geq 2 \;\; i = 1, \cdots, n$, $n-1$ natural numbers $i_1 \in \{1, 2, \cdots, b_1\}, \cdots, i_{n-1} \in \{1, 2, \cdots, b_n\}$ and n boolean variables $k_1, \cdots, k_n \in \{0, 1\}$:

Definition V.1

chronologically preordered temporal structure:

$$\begin{aligned} &structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots; t_{n,(i_{n-1})}, b_n, k_n) \; := \\ &\qquad Split(\cdots Split(Split(\mathbb{R}, \Omega(t_1), b_1), \Omega_{t_{2,(i_1)}} t_{2,(i_1)}, b_2) \cdots, [\Omega_{t_{n,(i_{n-1})}}, b_n)) \end{aligned} \tag{5.4}$$

where:

$$\Omega(t_1) := \begin{cases} [t_1, +\infty), & \text{if } k_1 = 0; \\ (-\infty, t_1], & \text{if } k_1 = 1 \end{cases} \tag{5.5}$$

$$\Omega(t_2) := \begin{cases} [t_{2,(i_1)}, +\infty_{(i_1)}), & \text{if } k_2 = 0; \\ (-\infty_{(i_1)}), t_{2,(i_1)}], & \text{if } k_2 = 1 \end{cases} \tag{5.6}$$

$$\vdots$$

$$\Omega(t_n) := \begin{cases} [t_{n,(i_{n-1})}, +\infty_{(i_{n-1})}), & \text{if } k_n = 0; \\ (-\infty_{(i_{n-1})}, t_{n,(i_{n-1})}], & \text{if } k_n = 1 \end{cases} \tag{5.7}$$

The elements of $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots; t_{n,(i_{n-1})}, b_n, k_n)$ with $k_i = 0$ will be called *time-divisions* while the elements of $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots; t_{n,(i_{n-1})}, b_n, k_n)$ with $k_i = 1$ will be called *time-stickings*.

Remark V.1

The terminology adopted in the definition V.1 is owed to the fact that the multiple application of definition II.3 induces on $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$ the chronological relation $\preceq_{chronological}$ that, by the multiple application of the proposition I.1, is a preordering.

Remark V.2

Let us remark that the iterated adoption of the definition II.2 induces a topology on $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$ that we will denote by $\mathcal{T}(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$.

Definition V.2

graph of $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$*:*

the oriented graph $graph[structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)]$ obtained by $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$

- replacing each *time-division* $\langle t_{i-1,(j)} \cdots , t_{i,(j)}) \cup \cup_{k=1}^{b_i} [t_{i,(k)}, t_{i+1,(k)} \rangle$ with a vertex t_i having one entering edge corresponding to the interval $\langle t_{i-1,(j)} \cdots , t_{i,(j)})$ and having b_i exiting edges corresponding to the intervals $[t_{i,(1)}, t_{i+1,(1)} \rangle, \cdots , [t_{i,(b_i)}, t_{i+1,(b_i)} \rangle$
- replacing each *time-sticking* $\cup_{k=1}^{b_i} \langle t_{i-1,(k)}, t_{i,(k)}] \cup (t_{i,(j)}, t_{i+1,j} \rangle$ with a vertex t_i having b_i entering edges corresponding to the intervals $\langle t_{i-1,(1)}, t_{i,(1)}], \cdots , \langle t_{i-1,(b_i)}, t_{i,(b_i)}]$ and having one exiting edge corresponding to the interval $(t_{i,(j)}, t_{i+1,j} \rangle$

where we have denoted by "$\langle$" a parenthesis that can be a "(" or a "[" while we have denoted by "$\rangle$" a parenthesis that can be a ")" or a "]".

Remark V.3

Such as definition V.1 is a generalization of definition IV.1 so definition V.2 is a generalization of definition IV.2.

As we have already done in the remark IV.3 as to temporal trees and their graphs we remark here that $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$ and its graph $graph[structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)]$ are two distinct mathematical notions.

Remark V.4

Let us observe that, taking into account the orientation of edges, $graph[structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)]$ doesn't contain loops.

Given a map $f \in C^{\infty}(\mathbb{R})$, an initial time $t_{in} \in structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$ and a real number $x_{in} \in \mathbb{R}$:

Proposition V.2

The Cauchy problem:

$$\dot{x} = f(x) \tag{5.8}$$

$$x(t_{in}) = x_{in} \tag{5.9}$$

is well-posed i.e. it admits one and only one solution
$\tilde{x}(t) : (structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n, b_n), \mathcal{T}(t_1, b_1; t_{2,(i_1)}, b_2; \cdots ; t_{n,(i_{n-1})}, b_n)) \mapsto (\mathbb{R}, +, \cdot)$

PROOF:

The thesis immediately follows by Proposition IV.1. ■

VI. THE INTRODUCTION OF CHRONOLOGICAL CIRCULARITIES IN THE TOY MODEL

In this section we will consider topologies of time giving rise to causal circularities.
Let us start from the simplest case:
given the topological space $(\mathbb{R}, T_{natural}(\mathbb{R}))$ and two points $t_1, t_2 \in \mathbb{R} : t_1 < t_2$:

Definition VI.1

identification of t_1 *and* t_2 *on* $\mathbb{R}$*:*

$$Identification(\mathbb{R}; t_1, t_2) \; := \; \frac{(\mathbb{R}, T_{natural}(\mathbb{R}))}{\sim_{t_1,t_2}} \tag{6.1}$$

where $\sim_{t_1,t_2}$ is the equivalence relation over $\mathbb{R}$:

$$a \sim_{t_1,t_2} b \; := \; \exists n \in \mathbb{Z} : b = a + n(t_2 - t_1) \tag{6.2}$$

Let us consider in particular the case $Identification(\mathbb{R}; 0, 2\pi)$ that is nothing but the circle S^1 endowed with the induced quotient topology.

Given $f \in C^{\infty}(\mathbb{R})$, $t_{in} \in \frac{\mathbb{R}}{\sim_{0,2\pi}}$ and $x_{in} \in \mathbb{R}$ as usual we will say that the Cauchy-problem:

$$\dot{x} = f(x) \tag{6.3}$$

$$x(t_{in}) = x_{in} \tag{6.4}$$

is well defined if and only it has one and only one solution $\tilde{\tilde{x}}(t) : Identification(\mathbb{R}; 0, 2\pi) \mapsto (\mathbb{R}, +, \cdot)$.

Denoted by $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ the solution of eq.6.3, eq. 6.4 for ordinary real time we have clearly that:

Proposition VI.1

The Cauchy problem of eq. 6.3, eq. 6.4 is well-defined if and only if:

$$(t_1 \sim_{0,2\pi} t_2 \; \Rightarrow \; \tilde{x}(t_1) = \tilde{x}(t_2)) \;\; \forall t_1, t_2 \in \mathbb{R} \tag{6.5}$$

in which case the solution is:

$$\tilde{\tilde{x}}([t]_{\sim_{0,2\pi}}) \; := \; \tilde{x}(t) \;\; \forall t \in \mathbb{R} \tag{6.6}$$

PROOF:

The thesis immediately follows by the definition VI.1 ■

Let us then consider the introduction of causal circularities in the topological space $(structure(0,2,0), \mathcal{T}(0,2,0))$; at this purpose let us identify the point $-\pi \in (-\infty, 0)$ and the point $+\pi_{(1)} \in [0_{(1)}, +\infty_{(1)})$ through the following:

Definition VI.2

identification of $-\pi$ *and* $+\pi_{(1)}$ *on* $structure(0,2,0)$*:*

$$Identification[structure(0,2,0); -\pi, +\pi_{(1)}] \; := \; \frac{(structure(0,2,0), \mathcal{T}(0,2,0))}{\sim_{-\pi,+\pi_{(1)}}} \tag{6.7}$$

where $\sim_{-\pi,+\pi_{(1)}}$ is the equivalence relation over $structure(0,2,0)$:

$$a \sim_{-\pi,+\pi_{(1)}} b \; := \; \exists n \in \mathbb{Z} : b = a + n[\pi_{(1)} - (-\pi)] \tag{6.8}$$

Definition VI.3

graph of $Identification[structure(0,2,0); -\pi, +\pi_{(1)}]$*:*

$graph\{Identification[structure(0,2,0); -\pi, +\pi_{(1)}]\}$:= the graph obtained from $graph[structure(0,2,0)]$ by the identification of the edge containing $-\pi$ and the edge containing $+\pi_{(1)}$.

Given $f \in C^{\infty}(\mathbb{R})$, $t_{in} \in \frac{structure(0,2,0)}{\sim_{-\pi,+\pi_{(1)}}}$ and $x_{in} \in \mathbb{R}$ as usual we will say that the Cauchy-problem:

$$\dot{x} = f(x) \tag{6.9}$$

$$x(t_{in}) = x_{in} \tag{6.10}$$

is well defined if and only it has one and only one solution $\tilde{\tilde{x}}(t) : Identification[structure(0,2,0); -\pi, +\pi_{(1)}] \mapsto (\mathbb{R}, +, \cdot)$.

Denoted by $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ the solution of eq. 6.9, eq. 6.4 for ordinary real time we have clearly that:

Proposition VI.2

The Cauchy problem of eq. 6.9, eq. 6.10 is well-defined if and only if:

$$a \sim_{-\pi,\pi_{(1)}} b \;\Rightarrow\; \tilde{x}(a) = \tilde{x}(b) \tag{6.11}$$

in which case the solution is:

$$\tilde{\tilde{x}}([t]_{\sim_{-\pi,+\pi_{(1)}}}) \;:=\; \tilde{x}(t) \;\; \forall t \in \mathbb{R} \tag{6.12}$$

PROOF:

The thesis immediately follows by the definition VI.2 ■

Let us now consider a chronologically preordered structure $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$ with $k_1 := 1$ and $k_n := 0$ so that the graph $graph\{structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)\}$ has b_1 input edges and b_n output edges.

Given $t_{A,(i)} \in (-\infty_{(i)}, t_{1,(i)}), i \in \mathbb{N}_+ : i \leq b_1$ and $t_{B,(j)} \in (t_{n,(j)}, +\infty_{(j)}), j \in \mathbb{N}_+ : j \leq b_1$:

Definition VI.4

identification of $t_{A,(i)}$ *and* $t_{B,(j)}$ *on* $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$*:*

$$Identification[structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n); t_{A,(i)}, t_{B,(j)}] \;:=\; \frac{(structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n), \mathcal{T}(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n))}{\sim_{t_{A,(i)}, t_{B,(j)}}} \tag{6.13}$$

where $\sim_{t_{A,(i)}, t_{B,(j)}}$ is the equivalence relation over $structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)$:

$$a \sim_{t_{A,(i)}, t_{B,(j)}} b \;:=\; \exists n \in \mathbb{Z} : b = a + n(t_{B,(j)} - t_{A,(i)}) \tag{6.14}$$

Given $f \in C^{\infty}(\mathbb{R})$, $t_{in} \in \frac{structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n)}{\sim_{t_{A,(i)}, t_{B,(j)}}}$ and $x_{in} \in \mathbb{R}$ as usual we will say that the Cauchy-problem:

$$\dot{x} = f(x) \tag{6.15}$$

$$x(t_{in}) = x_{in} \tag{6.16}$$

is well defined if and only it has one and only one solution $\tilde{\tilde{x}}(t)$: $Identification[structure(t_1, b_1, k_1; t_{2,(i_1)}, b_2, k_2; \cdots ; t_{n,(i_{n-1})}, b_n, k_n); t_{A,(i)}, t_{B,(j)}] \mapsto (\mathbb{R}, +, \cdot)$.

Denoted by $\tilde{x}(t) : \mathbb{R} \mapsto \mathbb{R}$ the solution of eq. 6.15, eq. 6.16 for ordinary real time we have clearly that:

Proposition VI.3

The Cauchy problem of eq. 6.15, eq. 6.16 is well-defined if and only if:

$$a \sim_{t_{A,(i)}, t_{B,(j)}} b \;\Rightarrow\; \tilde{x}(a) = \tilde{x}(b) \tag{6.17}$$

in which case the solution is:

$$\tilde{\tilde{x}}([t]_{\sim_{t_{A,(i)}, t_{B,(j)}}}) \;:=\; \tilde{x}(t) \;\; \forall t \in \mathbb{R} \tag{6.18}$$

PROOF:

The thesis immediately follows by the definition VI.4 ■

We can then suppose to perform more than an identification, resulting in the following:

Definition VI.5

temporal structure:

any topological space that can be obtained applying to a chronologically preordered temporal structure a finite number of identifications

VII. CONCLUSIONS LEARNT FROM THE TOY MODEL

The analysis of the toy model performed in the previous sections taught us that:

- as soon as the topological structure of time is chronologically preordered (so that the quotient with respect to such a preordering is a partial, and in the occurring case actually linear, ordering):
  - any single initial condition for the system represented by the model is the cause of its state at chronologically later times and is the effect of its state at chronologically previous times; parallel timelines are physically identical.
  - multiple initial conditions at any time-division point are the cause of the state at chronologically later times and are the effect of the state at chronologically previous times if and only if they are identical, once again leading to physically identical parallel timelines; if such a consistency condition is not satisfied no causation occurs.
- if chronological circularities are introduced in the topological structure of time, causation occurs if and only if suitable consistency conditions hold; in this case no mathematical contradiction arises from the existence of chronology violation [5]: in such a situation an event A may be both the cause and the effect of an event B without logical inconsistencies.

[5] It should be remarked that in the toy model the existence of chronological violation may be defined as the condition that $\preceq_{chronological}$ fails to be a preordering; hence, with this respect, the nonrelativistic toy model differs from the general relativistic situation in which, as stated by the proposition I.1 , the chronological relation is still a preordering over the chronology violating set failing to be a partial ordering.

VIII. VISSER'S MULTIHISTORY APPROACH TO TIME-TRAVEL PARADOXES

The first mathematical result interrelating the removal of the assumption that space-time, as a topological space, has to be Hausdorff and time-travel has been given (implicitly) by Petr Hajicek in the Theorem 4 of [16] that implies the existence, on a non Hausdorff space time (M, g_{ab}), of a deep interrelation between the presence of chronology violations, i.e. the situation in which $V_{chronology}(M, g_{ab}) \neq \emptyset$, and the presence of time-like bifurcating paths.

Of particular interest is Hajicek's analysis about Cauchy problems on non-Hausdorff topological spaces and bifurcating solutions that we will analyze, again, in a simple model.

Given a non-Hausdorff topological space (S, T) let us consider an alternative toy model, someway dual to the one discussed in the previous sections (and that we will call therefore the dual toy model), in that it is its configuration space to be an non Hausdorff topological space, being (S, T), while time is assumed to be described by the topologically trivial real line $\mathbb{R}$. Let us suppose that such a dynamical system is ruled by the differential equation $\dot{x} = f(x)$ where f is a continuous function on (S, T).

Given $t_{in} \in \mathbb{R}$ and $x_{in} \in S$ Hajicek analyzes the situation in which the Cauchy problem:

$$\dot{x} \ = \ f(x) \tag{8.1}$$

$$x(t_{in}) \ = \ x_{in} \tag{8.2}$$

has two solutions α_1 and α_2 bifurcating at a time $t_{bif} \in \mathbb{R} \ : \ t_{bif} > t_{in}$ (see appendix B).

According to our terminology such a dynamical system is still deterministic though no causal relationships exist between its states at different times.

Giving up the assumption that space-time has to be Hausdorff doesn't allow, by itself , to furnish a mathematical solution to the so-called time-travel paradoxes belonging to Matt Visser's option "Radical Rewrite Conjecture" [14], i.e. allowing changes of the past by a free-will's owner.

An example of how this can be performed adding some other ingredient to the permission of non-Hausdorff space-times has been given by Matt Visser:

given a solution (M, g_{ab}) of Einstein's equation such that $V_{chronoloy}(M, g_{ab}) \neq \emptyset$ Matt Visser's multihistory approach to the emerging so-called time travel paradoxes may be formalized as the addiction to General Relativity of the following:

AXIOM VIII.1

If a free-will's owner changes the past in $p \in V_{chronoloy}(M, g_{ab})$ then the following phase transition of the Universe occurs:

$$M \ \rightarrow \ Split(M, \overline{J^{+}(p)}, 2) \tag{8.3}$$

The analogous of axiom VIII.1 for our non relativistic toy model would consists in the following:

AXIOM VIII.2

If a free-will's owner changes the past at time $t_1 \in X$ then the following phase transition in the topological structure of time occurs:

$$X \ \rightarrow \ Split(X, t_1, 2) \tag{8.4}$$

where X is the temporal structure describing time before the action of the free-will's owner.

Let us call x_1 the value that $x(t_1)$ would have had if the free-will's owner hadn't changed the past at t_1.

So the following generalized Cauchy problem faces us:

$$\dot{x} \ = \ f(x) \tag{8.5}$$

$$x(t_{1,(1)}) \ = \ x_1 \tag{8.6}$$

$$x(t_{1,(2)}) \ = \ x_2 \tag{8.7}$$

where $x_2 \neq x_1$ is the new initial condition chosen by the free-will's owner.

By the conclusions summarized in the section VII it follows that the changes of the past break any causational relation.

IX. WARNING

A slightly extended version of this paper (containing figures and some example) is available at the author's homepage http://www.gavrielsegre.com

APPENDIX A: PREORDERINGS

In this appendix we will briefly recall the basic notions concerning relations over sets.
For the proof of the propositions, rather elementary, we demand to [21].
Given a set S:

Definition A.1

relation over S:

$$R \in \mathcal{P}(S \times S) \tag{A1}$$

where $\mathcal{P}(S) := \{X : X \subseteq S\}$ is the power-set of S.
Given a relation R over S let us introduce the useful notation:

$$xRy \ := \ (x, y) \in R \tag{A2}$$

Let us then introduce the following basic:

Definition A.2

R is a preordering:

- $$xRx \ \ \forall x \in S \tag{A3}$$

- $$(x_1 R x_2 \ \wedge \ x_2 R x_3 \ \Rightarrow \ x_1 R x_3) \ \ \forall x_1, x_2, x_3 \in S \tag{A4}$$

Definition A.3

equivalence relation over S:

a preordering $\sim$ over S such that:

$$(x_1 \sim x_2 \ \Rightarrow \ x_2 \sim x_1) \ \ \forall x_1, x_2 \in S \tag{A5}$$

Given an equivalence relation $\sim$ over S and $x \in S$:

Definition A.4

equivalence class of x w.r.t. $\sim$:

$$[x]_{\sim} \ := \ \{y \in S \ : \ x \sim y\} \tag{A6}$$

Definition A.5

quotient of S w.r.t. $\sim$:

$$\frac{S}{\sim} \ := \ \{[x]_{\sim}, x \in S\} \tag{A7}$$

Given a preordering $\preceq$ over S and $x_1, x_2 \in S$:

Definition A.6

$$x_1 \sim_{\preceq} x_2 \ := \ x_1 \preceq x_2 \ \wedge \ x_2 \preceq x_1 \tag{A8}$$

Proposition A.1

$$\sim_{\preceq} \text{ is an equivalence relation over S}$$

Definition A.7

partial ordering over S:

$$\text{a preordering } \preceq \text{ over S such that:}$$

$$(x_1 \sim_{\preceq} x_2 \;\Rightarrow\; x_1 = x_2) \;\; \forall x_1, x_2 \in S \tag{A9}$$

Definition A.8

linear ordering over S:

$$\text{a partial ordering } \preceq \text{ over S such that:}$$

$$(x_1 \preceq x_2 \;\vee\; x_2 \preceq x_1) \;\; \forall x_1, x_2 \in S \tag{A10}$$

APPENDIX B: NON HAUSDORFF TOPOLOGICAL SPACES

In this appendix we will briefly recall the basic notions concerning topological spaces with a particular emphasis to non Hausdorff ones.

For the proof of the propositions, rather elementary, we demand to [19].

Given a set S:

Definition B.1

topology over S:
$T \in \mathcal{P}(S)$:

-
$$\emptyset, S \in T \tag{B1}$$

-
$$O_1, O_2 \in T \;\Rightarrow\; O_1 \cap O_2 \in T \tag{B2}$$

-
$$O_i \in T \, \forall i \in I \;\Rightarrow\; \cup_{i \in I} O_i \in T \tag{B3}$$

where $\mathcal{P}(S) := \{X : X \subseteq S\}$ is the power-set of S and where I is an arbitrary index set of arbitrary cardinality.
We will denote the set of all the topologies over S by TOP(S).
Let us observe first of all that:

Proposition B.1

- discrete topology over S :=

$$T_{discrete}(S) := \mathcal{P}(S) \in TOP(S) \tag{B4}$$

- indiscrete topology over S :=

$$T_{indiscrete}(S) := \{\emptyset, S\} \in TOP(S) \tag{B5}$$

Given $T_1, T_2 \in TOP(S)$:

Definition B.2

T_1 is coarser than T_2 (T_2 is finer than T_1)

$$T_1 \preceq T_2 \;:=\; T_1 \subseteq T_2 \tag{B6}$$

Given $T \in TOP(S)$ and $x_1, x_2 \in S$

Definition B.3

$$x_1 \curlyvee x_2 \;:=\; O_1 \cap O_2 \neq \emptyset \;\; \forall O_1, O_2 \in T \;:\; x_1 \in O_1 \;\wedge\; x_2 \in O_2 \tag{B7}$$

Definition B.4

T is Hausdorff:

$$\neg(x_1 \curlyvee x_2) \;\; \forall x_1, x_2 \in S \tag{B8}$$

We will denote the set of all the Hausdorff topologies over S by $TOP_{Hausdorff}(S)$.

Proposition B.2

- $\preceq$ is a partial ordering over TOP(S)
- $$T_{indiscrete}(S) \preceq T \ \ \forall T \in TOP(S) \tag{B9}$$
- $$T \preceq T_{discrete}(S) \ \ \forall T \in TOP(S) \tag{B10}$$
- $$T_{indiscrete}(S) \notin TOP_{Hausdorff}(S) \tag{B11}$$
- $$T_{discrete}(S) \in TOP_{Hausdorff}(S) \tag{B12}$$
- $$T_1 \in TOP_{Hausdorff}(S) \wedge T_1 \preceq T_2 \ \Rightarrow \ T_2 \in TOP_{Hausdorff}(S) \tag{B13}$$

Definition B.5

topological space:
a couple (S, T):

- S is a set
- $T \in TOP(S)$

Given a topological space (S, T) the elements of T are called the open sets of (S, T).

Definition B.6

(S, T) is Hausdorff:

$$T \in TOP_{Hausdorff}(S) \tag{B14}$$

Given $A \subset S$:

Definition B.7

A is closed:

$$S - A \in T \tag{B15}$$

Definition B.8

closure of A:

$$\bar{A} := \text{the smallest closed set containing A}$$

Definition B.9

interior of A:

$$A^{\circ} := \text{the largest open set contained in A}$$

Definition B.10

boundary of A:

$$\partial A := \bar{A} - A^{\circ} \tag{B16}$$

Given $\{O_i\}_{i \in I} : O_i \in T \, \forall i \in I$:

Definition B.11

$\{O_i\}_{i \in I}$ *is a base of* (S,T)*:*

$$\forall O \in T \; \exists I' : O = \cup_{i \in I'} O_i \tag{B17}$$

Clearly a base in a topological space individuates univocally the underlying topology.

Definition B.12

natural topology over $\mathbb{R}$*:*

$$T_{natural}(\mathbb{R}) \in TOP(\mathbb{R}) \; : \{(a,b), a, b \in \mathbb{R} : a < b\} \text{ is a base of } (\mathbb{R}, T_{natural}(\mathbb{R})) \tag{B18}$$

Proposition B.3

$$T_{natural}(\mathbb{R}) \in TOP_{Hausdorff}(S) \tag{B19}$$

We will assume from here and beyond that $\mathbb{R}$, as a topological space, is endowed with the natural topology $T_{natural}(\mathbb{R})$.

Given a topological space $X = (S,T)$ let us suppose to have an equivalence relation $\sim$ over S.

Definition B.13

quotient of X w.r.t. $\sim$*:*

$$\text{the topological space } \frac{X}{\sim} := (\frac{S}{\sim}, \frac{T}{\sim}) \text{ where } \frac{T}{\sim} := \{[O] : O \in T\}$$

Definition B.14

The equivalence relation $\sim$ *is Hausdorff:*

$$\frac{X}{\sim} \text{ is an Hausdorff topological space}$$

Given two equivalence relations $\sim_1$ and $\sim_2$ over S:

Definition B.15

intersection of $\sim_1$ *and* $\sim_2$*:*

$$\text{the equivalence relation } \sim_1 \wedge \sim_2 \text{ over S such that:}$$

$$x \sim_1 \wedge \sim_2 y \; := \; x \sim_1 y \wedge x \sim_2 y \tag{B20}$$

Then:

Proposition B.4

$$\sim_1 \; Hausdorff \wedge \; \sim_2 \; Hausdorff \; \Rightarrow \sim_1 \wedge \sim_2 Hausdorff \tag{B21}$$

Given two topological spaces (S_1, T_1) and (S_2, T_2) and a map $f : S_1 \mapsto S_2$:

Definition B.16

f is continuous:

$$f^{-1}(O) \in T_1 \;\; \forall O \in T_2 \tag{B22}$$

Given a topological space (S, T):

Definition B.17

path in (S,T):

$$\alpha : (\mathbb{R}, T_{natural}(\mathbb{R})) \mapsto (S, T) \text{ continuous} \tag{B23}$$

Given two path α_1 and α_2 in (S,T) and a number $t_{bif} \in \mathbb{R}$:

Definition B.18

α_1 and α_2 bifurcate at t_{bif}:

$$\alpha_1(t) = \alpha_2(t) \; \forall t \in (-\infty, t_{bif}) \tag{B24}$$

$$\alpha_1([t_{bif}, +\infty)) \cap \alpha_2([t_{bif}, +\infty]) \; = \; \emptyset \tag{B25}$$

Proposition B.5

$$\exists \, \alpha_1 \text{ and } \alpha_2 \text{ paths in (S,T) bifurcating at } t_{bif} \in \mathbb{R} \; \Rightarrow \; \text{(S,T) is not Hausdorff}$$

PROOF:

By definition B.18 $\alpha_1(t_{bif}) \curlyvee \alpha_2(t_{bif})$. ■

APPENDIX C: DIFFERENTIAL CALCULUS ON NON HAUSDORFF TOPOLOGICAL SPACES

Given a set S which is the minimal suppletive mathematical structure $\mathcal{T}$ through which we have to endow S in order that on $(S, \mathcal{T})$ differential calculus may be defined ?

To answer such a deep question goes far beyond the goals of this paper.

For our purposes it will be enough to observe that as to the definition of limits, a metric structure is a too strong requirement since the induced topology is Hausdorff; contrary the assignment of a topology $\mathcal{T}$ (not necessarily Hausdorff) is sufficient:

given a map $f : S \mapsto S$ on the topological space $(S, \mathcal{T})$ and two point $x_1, x_2 \in S$:

Definition C.1

f tends to x_2 as x tends to x_1:

$$\lim_{x \rightarrow x_1} f(x) = x_2 \ := \ \forall O_2 \in \mathcal{T} : x_2 \in O_2 \ \exists O_1 \in \mathcal{T} \ : \ x_1 \in O_1 \ \wedge \ (f(x) \in O_2 \forall x \in O_1) \tag{C1}$$

In a similar way, given two sets S_1 and S_2, a map $f : S_1 \mapsto S_2$ and two point $x_1, \in S_1$ and $x_2, \in S_2$ the assignment of a topology $\mathcal{T}_1$ on S_1 and of a topology $\mathcal{T}_2$ on S_2 is sufficient to define limits:

Definition C.2

f tends to x_2 as x tends to x_1:

$$\lim_{x \rightarrow x_1} f(x) = x_2 \ := \ \forall O_2 \in \mathcal{T}_2 : x_2 \in O_2 \ \exists O_1 \in \mathcal{T}_1 \ : \ x_1 \in O_1 \ \wedge \ (f(x) \in O_2 \forall x \in O_1) \tag{C2}$$

As to the definition of the derivative of f the assignment of two topologies on domain and codomain is not sufficient.

It is in fact necessary to require that S_2 has the algebraic structure of being a field w.r.t. two internal binary operations $+$ and $\cdot$.

One can then introduce the following:

Definition C.3

derivative of f in x_1:

$$f'(x_1) \ := \ \lim_{x \rightarrow x_1} \frac{f(x) - f(x_1)}{x - x_1} \tag{C3}$$

Clearly the definition C.3 may be iterated in order to define derivatives of of any order.

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